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VISUALIZATION OF PHASE OBJECTS BY HILBERT TRANSFORM

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VISUALIZATION OF PHASE OBJECTS BY HILBERT TRANSFORM

Y. Belvaux and J. C. Vareille[1]

ABSTRACT. The Hilbert transform allows small phase shifts to be clearly exhibited. In the present article we study the properties of images obtained by Hilbert transform and also their development as a function of different experimental parameters. The Hilbert transform is obtained by placing a "phase knife" in the spectral plane of a double diffraction setup. In addition, we examine the precision with which the phase knife must be placed and centered in the spectral plane and the effect of width of source used. Experimental results are provided confirming the theoretical study.

INTRODUCTION

There are a great many methods of displaying phase objects, each one having its own field of application. However, it can be said, as a general rule, that the study of wide objects with small phase shifts and slow variations is difficult whatever method is used. /149*

Some devices, such as the Twyman-Green method and interferometry in polarized light, use two-wave interferences [1, 2, 3]. The methods of interferometry using polarized light are quite sensitive. However, in the case before us here involving slow phase variations with wide objects, their use is limited.

Another type of method uses the properties of double diffraction setups [8] such as phase contrast, interferometer photography or the Foucault method [2, 4, 5]. The Hilbert transform method [6, 13] belongs to this type.

Interferometer photography and phase contrast introduce into the spectral plane of a double diffraction setup [7, 8] a band or a capsule which is

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*Numbers in the margin indicate pagination in the foreign text.

either opaque (interferometer photography) or absorbing and phase shifting (phase contrast). There is then a "skip zone" for low spatial frequencies.

In order to study the distortions at low frequencies, it is possible to seek to use a phase knife contrast method. However, disturbing diffraction fringes appear. In practice, the Foucault method [9, 10] is used almost exclusively.

The Foucault method allows a very clear visualization of the wave surface when phase shifts are great and when it is possible to keep to one geometric argument [11]. In the case of small phase shifts, diffraction should be taken into account and the Hilbert transform then comes into the calculations as a natural consequence. We can prove (§1-6) that the Foucaultgram is in reality a combination of the object and the Hilbert transform of this object. It seems natural then to try for relief from the object at the same time keeping only the "effective" part of the phenomenon. Moreover, the Hilbert transform (T. H.) can be implemented using a source of white light [12]. /150

The purpose of this article is to show the properties of images obtained by the Hilbert transform and also their development as a function of different parameters.

The Hilbert transform is obtained by placing a "phase shift" in the spectral plane of a double diffraction setup. In the following pages we study the precision with which the phase knife must be prepared, the precision with which it must be centered in the spectral plane, as well as the effect of the width of source used.

Experimental results confirming the theoretical studies are given.

1. The Hilbert Transform in Optics

1.1. Definition of the Hilbert Transform

The Hilbert transform of an f(x) function of a variable x is mathematically defined by:

$$f_H(x) = \frac{1}{\pi}\,\text{v. p.}\int_{-\infty}^{+\infty} \frac{f(y-x)}{y}\,dy \qquad (1)$$

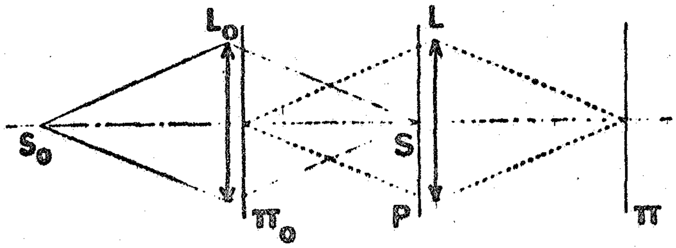
v.p. meaning "principal value" in the Cauchy sense.

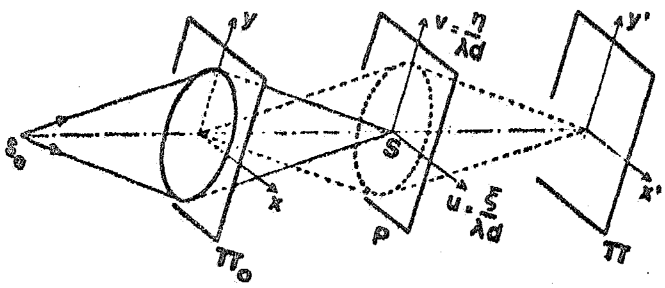
The expression can also be written, in the sense of distribution:

$$f_H(x) = -\frac{1}{\pi} f(x) * \frac{1}{x}$$

where * means convolution product.

Such a convolution product can be achieved very simply by using a double diffraction setup with coherent optics (Figure 1).





Figures 1 and 2. Flow Chart. The filter is placed within the spectral plane P.

A lens L_0 forms at S the image of a focused source S_0. The plane P perpendicular to the axis of the system at S is the spectral plane. A second lens L forms at Π the image of object plane Π_0.

When the function f(x) to be processed is introduced into the plane Π_0 (for example, in the form of a photographic transparency of transmittance with amplitude f(x), we obtain the Fourier transform or the F(u) spectrum of the object f(x) in plane P.

In the absence of a filter, a second diffraction gives in the plane Π the image of f(x) which is perfectly identical to the object, provided we disregard the bandpass limitation owing to the finite setting of lens L.

In order to obtain at Π the Hilbert transform of object f(x), it is enough, since the optical filtration is a linear filtration, to place in the spectral plane P a filter whose percussional response is 1/x, i.e., a filter whose transparency is the Fourier transform of 1/x, since this transformation is reciprocal. Such a filter exists and its transmittance in amplitude is:

$$H(u) = i \operatorname{sgn}(u) \tag{2}$$

with

$$\begin{aligned} \operatorname{sgn}(u) &= +1 \text{ for } u > 0 \\ &= -1 \text{ for } u < 0. \end{aligned}$$

It is also possible to express the sgn (u) function by using the Heavyside scale unit Γ(u) whose Fourier transform is known, and which provides two relations which will be useful later on:

$$\begin{aligned} \Gamma(u) &\xrightarrow{TF} \frac{1}{2}\left[\delta(x) - \frac{i}{\pi x}\right] \\ \Gamma(-u) &\xrightarrow{TF} \frac{1}{2}\left[\delta(x) + \frac{i}{\pi x}\right] \end{aligned} \tag{3}$$

TF = Fourier transform

where δ(x) depicts the Dirac measurement and it follows that

$$\operatorname{sgn}(u) = -\Gamma(-u) + \Gamma(u) \xrightarrow{TF} -\frac{i}{\pi x}.$$

It can be seen that since the solution of the Hilbert transform is a linear filtration, all the known properties of linear filtration are applicable. Especially, since the filter gain is with a unitary modulus, all the energy is found in the streak of light. Indeed, when f(x) has F(u) for the Fourier transform and fh(x) for the Hilbert transform:

$$\begin{aligned} \int_{-\infty}^{+\infty} |f_H(x)|^2\, dx &= \int_{-\infty}^{+\infty} |F(u) \operatorname{sgn}(u)|^2\, du \\ \int_{-\infty}^{+\infty} |F(u)|^2\, du &= \int_{-\infty}^{+\infty} |f(x)|^2\, dx. \end{aligned} \tag{4}$$

Therefore, we can see that, contrary to Foucault methods of interferometer /151
photography phase contrast, where a part of the energy is absorbed, the total incident energy is used here to form the image.

1.2. The Hilbert Transform Allows Visualization of Small Phase Shifts

Let us show, in a first example, how the Hilbert transform helps to visualize small phase shifts.

Let f(x) be an object characterized by a small sinusoidal phase variation.

$$f(x) = \exp\left(ia \cos \frac{2\pi x}{p}\right) \operatorname{rect} \frac{x}{D}$$

with $a \ll 1$.

Such an object is written disregarding terms of higher order:

$$f(x) = \left(1 + ia \cos \frac{2\pi x}{p}\right) \operatorname{rect} \frac{x}{D}$$

of which it can be shown that the Hilbert transform is:

$$f_H(x) = -ia \sin \frac{2\pi x}{p} + \frac{1}{\pi} \operatorname{Log} \left| \frac{x + \frac{D}{2}}{x - \frac{D}{2}} \right| \qquad (5)$$

D representing the width of the object field, i.e., in fact the diameter of the lens L_0 of Figure 1.

Except in the case of the points located very close to

$$x = \pm \frac{D}{2},$$

the last term of (5) can generally be disregarded and the illumination observed in the plane II is

$$E(x) = a^2 \sin^2 \frac{2\pi x}{p}.$$

The phase sine wave was transformed into an amplitude sine wave. Therefore, the optical solution of the Hilbert transform allowed the visualization of the sine wave phase object.

It should be noted that when the limited aperture of the lens L is taken into account, the amplitude in plane Π is the convolution of fH(x) through the diffraction spot d(x) of L; a response is observed in plane Π:

$$f'_H(x) = f_H(x) * d(x). \tag{6}$$

The "infinite peaks" corresponding to $x = \pm D/2$ do not exist any longer. In the case of $x = \pm D/2$, we notice an accumulation of light which is a standard phenomenon in the Foucault type setup.

1.3. The Reduction Theorem

Before considering the case of objects more complex than the one in the preceding paragraph, it is advisable to show that, although optical setups are two-dimensional, it is correct to perform a one-dimensional argument. For this reason, we will see that the reduction theorem proven by Linfoot in the case of the Foucault method [10] is applied to any one-dimensional filter, particularly in the case of the Hilbert transform.

Let us reproduce in perspective (Figure 2) the flow chart of Figure 1 and disregard the limitation of the bandpass owing to the Lens L, i.e., that we assume this lens to have infinite size.

Let f(x,y) be the object function, the mechanism of the image formation is written:

$$f(x, y) \xrightarrow{TF} F(u, v) \xrightarrow{TF^{-1}} f(x, y) \tag{7}$$

u and v are the "spatial frequencies" in the spectral plane P and F(u,v) represents the Fourier transform of f(x,y).

If we introduce into plane P a one-dimensional filter whose transmittance in amplitude is a function of u alone H(u), the chart (7) becomes:

$$f(x, y) \xrightarrow{TF} F(u, v) \underbrace{\times\ H(u)}_{\text{filter}} \xrightarrow{TF^{-1}} f(x, y)_{x,y} * $$
$$* \, h(x)\, \delta(y) = f(x, y)_x * h(x) \tag{8}$$

The response is therefore a convolution product only dealing with the variable x (y becomes a parameter).

This constitutes the reduction theorem allowing, when one-dimensional screens are used, the correction of calculations with Fourier transforms to a single dimension.

The response obtained in plane x'y' is written:

$$r(x', y') = \int_{-\infty}^{+\infty} F(u, y')\, e^{i2\pi ux'}\, du \tag{9}$$

with

$$F(u, y') = \int_{-\infty}^{+\infty} f(x, y')\, e^{-i2\pi ux}\, dx\,. \tag{10}$$

Therefore, it can be seen that the value of the response r is only a function of the value of function f on a parallel line with the axis of the x values of y ordinate.

In the special case of the Hilbert transform, the filter

$$H(u) = \Gamma(u) - \Gamma(-u) \tag{11}$$

leads to a response:

$$r(x' y') = \int_{0}^{+\infty} F(u, y')\, e^{i2\pi ux'}\, du - \int_{-\infty}^{0} F(u, y')\, e^{i2\pi ux'}\, du\,. \tag{12}$$

We have assumed that the spectral plane P was not limited. This approximation is generally right. Nevertheless, it can lead to physical impossibilities (infinite illumination) when we consider points located on the edges of the object field defined by the diaphragm limiting the plane Π_0 of Figure 2. When such a case occurs, it can always be assumed that the spectral plane is in fact limited by a slit with width a, placed parallel to axis v (Figure 3). /152

We can then repeat the same proof, replacing H(u) by:

$$H'(u) = H(u) \times \operatorname{rect} \frac{u}{u_0} \tag{13}$$

where

$$\begin{cases} \mathrm{rect}\,\dfrac{u}{u_0} = 1 & \text{for } |u| \leqslant \dfrac{u_0}{2}; \quad \left(u_0 = \dfrac{a}{\lambda d}\right) \\ \quad = 0 & |u| > \dfrac{u_0}{2}. \end{cases}$$

The response is in this case:

$$r'(x',y') = r(x',y')*b(x') \tag{14}$$

where

$$b(x') = TF^{-1}\left[\mathrm{rect}\,\frac{u}{u_0}\right].$$

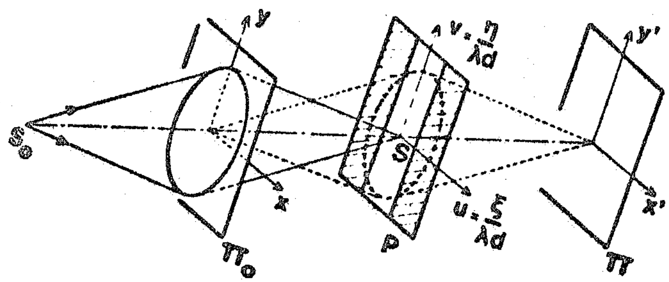


Figure 3. Limitation of the Spectral Plane by a Large Slit.

We are now going to be able to apply these results to the study of devices used to solve the Hilbert transform.

1.4. <u>The Hilbert Transform of an Object with Uniform Transmittance</u>

This case is important for in reality it shows location of the upper boundary of the stray light owing to diffraction present in the field during observation of any object.

Here, the object can be the aperture of diaphragm D limiting the spherical wave coming from the lens L_0. It can also be the mounting of a perfectly spherical mirror, in whose center of curvature the source S is set. The upper boundary of the stray light will be obtained, for when a phase object is introduced into the aperture of this diaphragm, the energy, in its image, is picked up on the mount obeying the rule of conservation of energy (4).

In the case of a circular diaphragm

$$f(x) = \text{rect}\,\frac{x}{D}$$

with

$$D = 2\sqrt{\frac{D_0^2}{4} - y^2}\,.$$

For an object of another shape, D will be a specific function of x, but the calculations to be done will remain the same.

Let $f(x) = \text{rect}\ x/D$ be the object function which we will assume to be centered on $x = 0$ for, since the Hilbert transform is linear and homogeneous, a transfer of the object leads to a simple transfer of the image.

The response in plane Π is:

$$R_1(x) = \frac{ia}{\lambda d}\left\{\text{rect}\,\frac{x}{D} * \left(-\frac{1}{\pi x}\right)\right\} * \frac{\sin px}{px} \qquad (15)$$

where

$$p = \frac{\pi a}{\lambda d}\,.$$

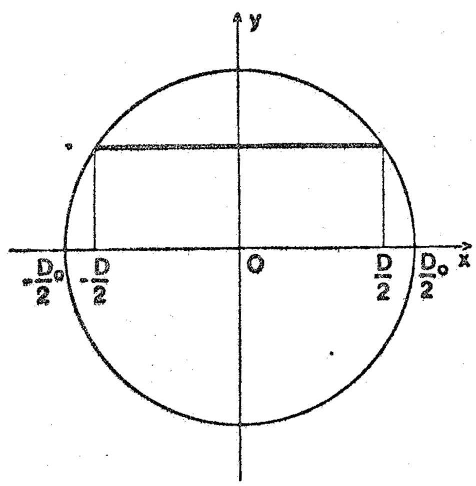


Figure 4. The Illumination Along a Straight Line with the Ordinate y Depends only on the Object Function Along This Straight Line.

Since the product of convolution is commutative and since the Hilbert transform of sin px/px [14] is known, (15) is written:

$$R_1(x) = \frac{i}{\pi} \operatorname{rect} \frac{x}{D} * \frac{\cos px - 1}{px}$$

which after integration suggests:

$$R_1(x) = \frac{i}{\pi} \left\{ \operatorname{Log} \left| \frac{x + \frac{D}{2}}{x - \frac{D}{2}} \right| - \operatorname{Ci}\left[p \left| x + \frac{D}{2} \right|\right] + \operatorname{Ci}\left[p \left| x - \frac{D}{2} \right|\right] \right\} \tag{16}$$

where Ci(x) represents the integral cosine of x defined for $x > 0$ by:

$$\operatorname{Ci}(x) = -\int_x^\infty \frac{\cos t}{t} \, dt.$$

We again find a standard result (5) which, when integral cosines /153
generally disregarded as soon as x is moved from ± D/2 are not taken into account, is reduced to:

$$R_1(x) = \frac{i}{\pi} \operatorname{Log} \left| \frac{x + \frac{D}{2}}{x - \frac{D}{2}} \right|.$$

The illumination in plane π is then:

$$E(x) = \frac{1}{\pi^2} \operatorname{Log}^2 \left| \frac{x + \frac{D}{2}}{x - \frac{D}{2}} \right|. \tag{17}$$

When x tends towards D/2, the development in series of Ci(x) to the vicinity of 0 brings

$$R_1(x) = \frac{i}{\pi} \left\{ \operatorname{Log} \left(p \left| x + \frac{D}{2} \right| \right) + \gamma \right\} \tag{18}$$

where y is the Euler constant.

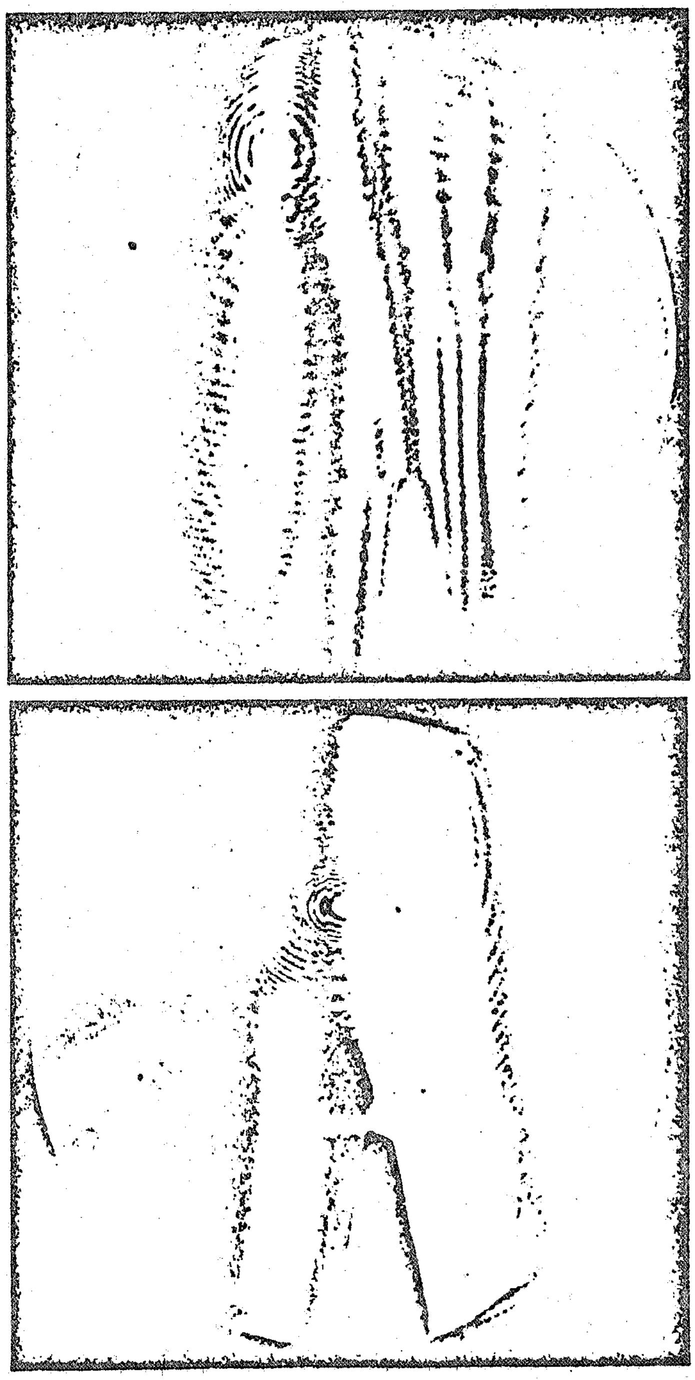


Figure 6. Observation of a Volume of Heated Air. a) Hilbert transform, b) Foucault method.

Special case. The perfectly transparent circular object. It is enough to take up $D = \sqrt{D^2/4-y^2}$ in (17). It can be seen that the isophotes are curves such that:

$$x + \sqrt{\frac{D_0^2}{4} - y^2} = k\left(x - \sqrt{\frac{D_0^2}{4} - y^2}\right).$$

These are ellipses with axis oy. We find, to approximately one additive constant, the result obtained by Linfoot [10] in the case of the Foucault method.

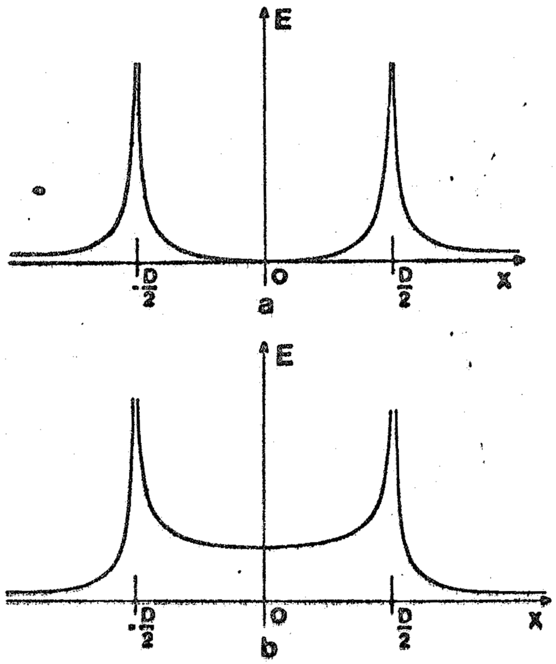


Figure 5. Illumination Obtained in the Case of a Perfect Lens (or Mirror). a) Hilbert transform, b) Foucault method.

1.5. Case of a Phase Bulge

It can be seen, according to the preceding paragraph, that the Hilbert transform of a cosine is a sine, i.e., that an even function is changed into an odd function and vice versa. This is very general and can be directly deduced from formula (1) defining the Hilbert transform. It follows that when we have an object constituted by a bulge with symmetrical phase, the latter will appear in the Hilbert transform in the shape of a dark line indicating the position of the maximum (or minimum) of the phase bulge. The localization is, in this case, more precise than in the Foucault method, where the phase crest corresponds to a zone of partial shadow (representation by glancing illumination). /154

In the case of an asymetrical phase bulge, the mathematical study is more complex. The dark minimum no longer corresponds exactly to the phase maximum, but differs very little from it. The photographs of Figure 6 show a volume of hot air observed by both methods: (a) Foucault method, (b) Hilbert transform.

It can be seen that the dark fringes are much more contrasted in the case of the Hilbert transform method.

1.6. Comparison with Other Visualization Methods

We plan in this paragraph to compare the Hilbert transform method to other usual methods of observation of phase distortions. Each method has in fact its field of application, and it is interesting to see where the Hilbert transform method is located relative to the others, for no method is indeed universal.

A first method for studying small phase objects consists of the two-wave interferometric method, using the Michelson interferometer, for instance. In the case of such an interferometer adjusted to a flat hue, the illumination is in the form:

$$E = 1 + \cos(\varphi(x) + \varphi_1)$$

φ_1 corresponding to the difference in operation between the two arms of the interferometer. The extreme cases which can be obtained correspond either to $\varphi_1 = 0$, $\varphi(x)$ being small, in which case we then practically have $E = 1$, in the whole field; or to $\varphi_1 = \pi/2$ in this case (corresponding to the adjustment with "black background" of the interferometer), the contrast is equal to $\varphi(x)$, i.e., very slight. The contrast is defined by:

$$\gamma = \frac{E_1 - E_2}{E_1}$$

where E_1 represents the illumination in the region of the image characterized by the phase distortion and E_2 the illumination of the background.

In order to observe such slight contrasts, the use of a Michelson interferometer assumes that we have reference wave surfaces, i.e., mirrors, defined with a precision clearly able to cope with the seriousness of the distortions to be revealed. It is not the same with differential polarization interferometers [1, 3]. The latter five excellent images when phase distortions have a relatively short period with regard to the size of the object field. Nevertheless, for objects with low spatial frequencies, it is easy to see that the conclusions are similar to those obtained in the case of the Michelson interferometer. As a matter of fact, the method is no longer applicable for it reveals the wave surface to be studied by its slopes.

The method by phase contrast consists in placing a small absorbing screen in the center of the spectrum of the object to be studied gives excellent results, mainly in microscopy, but cannot be applied to objects with very low spatial frequencies because of the finite size of the phase capsule set in the spectral plane. For the same reason, interferometer photography (opaque capsule) cannot be applied to this type of object.

In the case of objects with low spatial frequencies, the Foucault method is practically the only one used. We have therefore found it appropriate to compare the latter with the Hilbert transform method and we will see that, generally the Hilbert transform method is more sensitive than the Foucault method.

When objects with low frequencies are studied, a very small light source must be used. It could possibly be beneficial to use a laser. However, the presence of dusts and stray reflections in the system causes, considering the great temporal coherence of the laser, phenomena of interference, whose fringes in the observation plane considerably disturb the image observed.

It is preferable in this case to use standard sources which can advantageously use a wide spectrum [12].

In order to compare both methods, we will assume that the knife is centered on the diffraction spot which is usually the case, even with the Foucalt method, during the study of low frequency distortions. In the case of the Foucault method, the filter is then represented by the function $\Gamma(u)$ and the percussional response is, according to (3):

$$r(x) = \frac{1}{2}\left\{\delta(x) - \frac{i}{\pi x}\right\}$$

whence

$$f_F(x) = \frac{1}{2}\{f(x) + if_H(x)\} \tag{19}$$

fF represents the amplitude in the Foucaultgram.

Case of a sine-wave object. A sine-wave phase object which causes a small shift in aperture D of the object field is considered. The object function is written:

$$f(x) = \exp\left(i\varphi_0 \cos\frac{2\pi x}{p}\right)\operatorname{rect}\frac{x}{D}$$

the phase shifts being small $\varphi_0 \ll 1$

$$f(x) \simeq \left[1 + i\varphi_0 \cos\frac{2\pi x}{p}\right]\operatorname{rect}\frac{x}{D}$$

whose Hilbert transform is: /155

$$f_H(x) = -i\varphi_0 \sin\frac{2\pi x}{p} + \frac{1}{\pi}\operatorname{Log}\left|\frac{x + \frac{D}{2}}{x - \frac{D}{2}}\right| \tag{20}$$

which is written for points not located on the edge of the field:

$$f_H(x) \simeq i\varphi_0 \sin\frac{2\pi x}{p}.$$

The illumination in the image plane is then:

$$E(x) = \varphi_0^2 \sin^2 \frac{2\pi x}{p}. \tag{21}$$

The phase sinusoid was transformed into an amplitude sinusoid appearing with a modulation equal to 1, at least as long as the second term of the righthand side of expression (20) remains negligible, i.e., for points not located at the edge of the field. Since the object is periodic, the visibility of the phenomenon is preferentially characterized by the "modulation" defined as the ratio:

$$M = \frac{E_{max} - E_{min}}{E_{max} + E_{min}}$$

Insofar as the Foucault method is concerned, expression (19) allows obtaining the illumination directly:

$$E(x) \simeq \frac{1}{4}\left[1 + 2\varphi_0 \sin\frac{2\pi x}{p}\right]. \tag{22}$$

The modulation here is proportional to $2\varphi_0$ and no longer equal to 1 (we disregarded here the stray light of the system). It is this difference which constitutes the advantage of the Hilbert transform. The continuous background contributing to a lower modulation in the Foucault method is due to the term f(x) in expression (19), a term which disappears in the case of the Hilbert transform. In objection, it could be said that the Foucault method (22) provides a variation of illumination proportional to the phase φ_0, whereas the Hilbert transform (21) provides an illumination proportional to φ_0^2. Hence, we are brought back to the case of interferometer photography. Indeed, when very small phase shifts are considered, the contrast of the Foucault image will be very slight. Moreover, the analogy between the image obtained by the Hilbert transform and by interferometer photography is purely formal since:

a) by the method of interferometer photography the image of low frequency objects does not exist because of the finite size of the screen introduced into the spectral plane;

b) in the case of interferometer photography, the stray light owing to the diffraction by the edges of the opaque screen spreads over the whole field and reduces the contrast considerably [5] whereas in the Hilbert transform method, the stray light, or haze, is concentrated on the edges of the field in shape of a luminous ring.

It can be shown more precisely that, in the Hilbert transform, the gain in sensitivity comes with a decrease of the useful field, i.e., that the luminous ring edging the images is larger in the case of the Hilbert transform method than in the case of the Foucault method.

If consideration is only given the stray light of "diffraction" just studied and which is impossible to reduce, the Hilbert transform method offers an infinite sensitivity in the center of the field. In practice, there are scatterings inside optical parts as well as reflections on their surfaces. Consequently, a certain quantity of light is distributed, in a way which can be assumed to be uniform, in the image and contributes to lower the contrast (or modulation) observed.

When M is the smallest modulation perceptible by the observer and E_p the stray illumination, the smallest phase shift that the Hilbert transform method can detect is

$$\varphi_0 = \sqrt{\frac{2 E_p M}{1 - M}} \simeq \sqrt{2 E_p M} .$$

When we take, for example, M = 2% and $E_p = 10^{-4}$, it can be seen that the Hilbert transform method is 5 times more sensitive than the Foucault method.

Figure 5 shows, in the case of a perfectly transparent object, the illumination according to an horizontal diameter given:

a) by the Hilbert transform,

b) by the Foucault method with centered knife.

It can be seen beginning with (17) that, in the case of the Hilbert transform, only one third of the energy is located inside the geometric image; whereas the Foucault method leads to the opposite result, i.e., two-thirds of the energy inside the geometric image [10].

2. Practical Solution of the Hilbert Transform. The Real Knife

In practice, the phase knives used are not perfect. They are generally obtained by deposit in vacuo of a dielectric layer whose thickness is adjusted such that the phase shift is π for the working wavelength and such that the absorption is low. On the other hand, the edge of the phase knife does not always cut the diffraction figure exactly the middle. It should also be noted that the illumination source used is a slit parallel to the edge of the knife and has a specific width.

We plan in this part of the report to study the effects of various parameters.

2.1. General Expression of the Response Given by a Phase Filter /156

We shall consider here the completely general case of a filter formed by two juxtaposed spectral regions, not having the same optical thickness and not showing the same absorption property. Let $\tau \exp(i\theta)$ be the complex transmittance from one of the spectral regions with respect to the other. Such a filter, used as a phase knife, has a transmittance:

$$H(u) = \tau e^{i\theta} \Gamma(-u + u_0) + \Gamma(u - u_0)$$

where u_0 determines the position of the separation line of the spectral regions in the spectral plane. In this case expressions (3) become:

$$\Gamma(u - u_0) \xrightarrow{\mathrm{TF}} \frac{e^{-i2\pi u_0 x}}{2}\left\{\delta(x) - \frac{i}{\pi x}\right\} \tag{23}$$
$$\Gamma(-u + u_0) \xrightarrow{\mathrm{TF}} \frac{e^{-i2\pi u_0 x}}{2}\left\{\delta(x) + \frac{i}{\pi x}\right\}.$$

The percussional response of the filter is then:

$$r(x) = \frac{e^{-i2\pi u_0 x}}{2}\left\{(1 + \tau e^{i\theta})\,\delta(x) - \frac{i}{\pi x}(1 - \tau e^{i\theta})\right\}. \tag{24}$$

When such a filter is used to deal with an object function $f(x)$, the response of the system is:

$$R(x) = f(x)*r(x) \tag{25}$$

or again, when it is desired to take into account the bandpass, limited toward the higher frequencies (i.e., when we consider an object point located near a discontinuity)

$$R'(x) = f(x)*r(x)*d(x).$$

It can be seen that expression (24) covers all the possible cases of knife setups according to the values of θ and τ, particularly:

free pupil

$$\theta = 0 \quad \tau = 1$$

phase contrast

$$\theta = \pm\frac{\pi}{2} \quad \tau \leqslant 1$$

Hilbert transform

$$\theta = \pi \quad \tau = 1 \quad u_0 = 0$$

Foucault method

$$\tau = 0.$$

The expression (24) will enable us to study, as far as the Hilbert transform is concerned, what the effects are from:

-- a phase shift slightly different from π,

-- an absorption of the phase shifting region,

-- the decentralization of a filter in the object spectrum.

This last point will enable us to study the effect of the width of the illumination source.

2.2. Effect of the Absorption

It is assumed that one of the two regions of the filter shows a phase shift π and an absorption ε, i.e., a transmittance $\tau = 1 - \varepsilon$, the percussional response becomes

$$r(x) = \frac{1}{2}\left\{\varepsilon\,\delta(x) - \frac{i}{\pi x}(2-\varepsilon)\right\} \qquad (26)$$

whence the response for a function f(x)

$$R(x) = \tfrac{1}{2}\{\,\varepsilon f(x) + i(2-\varepsilon)f_H(x)\,\}$$

where $f_H(x)$ represents Hilbert transform of f(x).

There can then be seen in the image plane, an illumination

$$E(x) = \tfrac{1}{4}\{\,\varepsilon^2\,|f(x)|^2 + (2-\varepsilon)^2\,|f_H(x)|^2 + 2\,\mathrm{Re}\,[i\varepsilon(2-\varepsilon)f(x)f_H^*(x)]\,\} \qquad (27)$$

which takes, of course, the value $|f_H(x)|^2$ when $\varepsilon = 0$.

When the function f(x) is even or odd, real or purely imaginary, the illumination observed in the Hilbert transform $E = |f_H(x)|^2$ is then symmetrical in respect to x = 0 and this symmetry is not affected by the absorption of one of the regions of the filter, even when this absorption becomes total (Foucault).

In the case of the perfectly transparent object (§1.4), the intensity obtained will be depicted by a curve intermediary between the curves a) and b) of Figure 5. The intensity will no longer be equal to zero in the center of the field, but the presence of such an absorption will be hardly perceptible to the eye even when using photometric measurements, for it will possibly be merged with a haze of stray light.

The illuminations inside and outside the geometric image are respectively

$$E_{int} = \tfrac{1}{4}[\varepsilon^2 + (2-\varepsilon)^2\,|R_1(x)|^2]$$

$$E_{ext} = \tfrac{1}{4}(2-\varepsilon)^2\,|R_1(x)|^2 \qquad (28)$$

where $R_1(x)$ is the theoretical response (16)

$$R_1(x) = \frac{i}{\pi}\left[\text{Log}\left|\frac{x+\frac{D}{2}}{x-\frac{D}{2}}\right| - \text{Ci}\left(p\left|x+\frac{D}{2}\right|\right) + \text{Ci}\left(p\left|x-\frac{D}{2}\right|\right)\right] \qquad (29)$$

2.3. Effect of an Error in Phase Shift /157

When the phase shift differs slightly from π, $\theta = \pi - \alpha$ is granted and it is possible to write the percussional response (24) in the form

$$r(x) = \frac{1}{2}\left\{(1 - e^{-i\alpha})\,\delta(x) - \frac{i}{\pi x}(1 + e^{-i\alpha})\right\}$$
$$\simeq \frac{1}{2}\left\{i\alpha\delta(x) - \frac{i}{\pi x}(2 - i\alpha)\right\}$$

which leads, for an f(x) function, to the response

$$R(x) = \tfrac{1}{2}\{i\alpha f(x) + i(2 - i\alpha) f_H(x)\}$$

and the illumination observed in the image plane is

$$E(x) = \tfrac{1}{4}\{\alpha^2 ff^* + (4 + \alpha^2) f_H f_H^* + {} + 2\,\text{Re}\,[\alpha(2 + i\alpha) ff_H^*]\}. \qquad (30)$$

When f(x) is even or odd, real or purely imaginary, i.e., when the illumination observed for a phase shift strictly equal to π is symmetrical, it may be ascertained that this symmetry no longer exists when the phase shift differs from π.

As a matter of fact, the term from which we take the real part is then odd and not purely imaginary, for ff_H^* is odd.

It may be ascertained, therefore, that in the case of the perfectly circular object of §1.4, the image will no longer be symmetrical.

The illuminations observed inside and outside the geometric image will be respectively

$$E_{int} = \frac{1}{4}\left\{ \alpha^2 + (4 + \alpha^2)\,|\,R_1(x)\,|^2 + 2\,i\alpha^2\,R_1(x) \right\}$$
$$E_{ext} = \frac{1}{4}\left\{ (4 + \alpha^2)\,|\,R_1(x)\,|^2 \right\} \tag{31}$$

where $R_1(x)$ is still given by the expression (29).

It can be seen that the illumination in the halo located outside of the geometric image is, for its part, always symmetrical.

2.4. Off-centering and Width of the Source

First of all, it can be seen that, for questions of luminosity, in the double diffraction setup used, it is possible to replace the focused source S_0 by a slit parallel to the direction of the phase knife placed in the spectral plane. Indeed, according to the reduction theorem, or its adaptation made by us at the same time considering the spectral plane founded by a very elongate slit made parallel to the 0 axis, the illumination following a straight line located on the ordinate y and parallel to Ox (Figure 2), is only a function of the value of the object function following this straight line of ordinate y. When it is assumed that all the points of the source are incoherent, an illumination can be seen which is the sum of the illuminations owing to each small source component. Therefore, it can be seen that the same result is produced whether the system is illuminated with a focused source or with a linear source.

The elongation of the source in a direction parallel to Oy does not change anything concerning the results of filtration (in practice we must limit ourselves to a length such that the aberrations remain negligible). We will see what effect the width of this source has on the problem. For this reason, we will first study the effect of an off-centering in the case of an infinitely thin source; a simple integration will then enable us to study the effect of the width of this source.

2.4.1. Off-centering of the Source

Taken up first is the case of a perfect system illuminated by a homogeneous wave (for example, a perfect mirror working as its center.

Using again the calculation of §1.4 the equation (15) is written, in the case of a slanting wave with angle α (Figure 7)

$$R_1(x) = \frac{ia}{\lambda d}\left\{f(x)\,\mathrm{rect}\,\frac{x}{D} * \left(\frac{-1}{\pi x}\right)\right\} * \frac{\sin px}{px} \tag{32}$$

where f(x) represents the object wave. Here

$$f(x) = \exp\left(i\,\frac{2\,\pi\alpha x}{\lambda}\right) \tag{33}$$

α representing the slant of the wave, i.e., in fact, the position of the source.

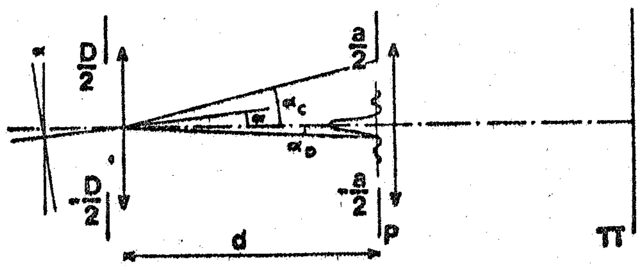


Figure 7. The Maximal Dip of the Object Wave Corresponds to α_c (Cutting Angle).

A similar calculation to the one already performed in §1.4 leads to the general expression of the response

$$R_1(x) = \frac{i}{\pi}\exp\left(i\,\frac{2\,\pi\alpha x}{\lambda}\right)[C - iS] \tag{34}$$

with

$$C = \frac{\mathrm{Ci}\,|\sigma + p|\left|x - \frac{D}{2}\right| + \mathrm{Ci}\,|\sigma - p|\left|x - \frac{D}{2}\right|}{2} - \frac{\mathrm{Ci}\,|\sigma + p|\left|x + \frac{D}{2}\right| + \mathrm{Ci}\,|\sigma - p|\left|x + \frac{D}{2}\right|}{2}$$

$$+\mathrm{Ci}\,|\sigma|\left|x + \frac{D}{2}\right| - \mathrm{Ci}\,|\sigma|\left|x - \frac{D}{2}\right|$$

$$S = \frac{\mathrm{Si}\,(\sigma + p)\left(x - \frac{D}{2}\right) + \mathrm{Si}\,(\sigma - p)\left(x - \frac{D}{2}\right)}{2} - \frac{\mathrm{Si}\,(\sigma + p)\left(x + \frac{D}{2}\right) + \mathrm{Si}\,(\sigma - p)\left(x + \frac{D}{2}\right)}{2} + \mathrm{Si}\,\sigma\left(x + \frac{D}{2}\right) - \mathrm{Si}\,\sigma\left(x - \frac{D}{2}\right)$$

where, as previously, Si and Ci mean integral sines and cosines and where

$$p = \frac{\pi a}{\lambda d} = \frac{2\,\pi\alpha_c}{\lambda}; \quad \sigma = \frac{2\,\pi\alpha}{\lambda} \tag{35}$$

the illumination received in observation plane Π is then:

$$E = \frac{1}{\pi^2}\{C^2 + S^2\}. \tag{36}$$

It can be seen that S and C are respectively even and odd functions of x and that, as a consequence, E takes again the same value when we change x into -x. The illumination in the field is symmetrical with respect to x = 0. This is shown by the curves of Figure 8 representing the theoretical illumination as a function of the source position (calculations performed on the Univac 1108 in collaboration of S. Slansky at the Faculty of Sciences of Orsay).

When x tends toward ± D/2, the term C takes an indeterminate value and must be replaced, at the same time using the relation Ci z = log z + γ by:

$$C' = \frac{1}{2}\,\mathrm{Log}\,\frac{(\sigma + p)(\sigma - p)}{\sigma^2} - \frac{\mathrm{Ci}\,|\sigma + p|\left|x + \frac{D}{2}\right| + \mathrm{Ci}\,|\sigma - p|\left|x + \frac{D}{2}\right|}{2} + \mathrm{Ci}\,|\sigma|\left|x + \frac{D}{2}\right|. \tag{37}$$

When σ tends toward zero we find again, as the amplitude, the expression (16) of §1.4.

When the bandpass is not greatly limited in the spectral plane, we have found that it was possible to consider it infinite provided that we exclude from the study the points in the direct vicinity of the edge of the field. The expression (34) is simplified and the response becomes:

$$\begin{aligned} R_1(x) = {} & \frac{i}{\pi} \exp\left(i \frac{2\pi\alpha x}{\lambda}\right) \times \\ & \times \left[\mathrm{Ci}\,|\sigma|\left|x + \frac{D}{2}\right| - \mathrm{Ci}\,|\sigma|\left|x - \frac{D}{2}\right|\right. \\ & \left. - i\left\{\mathrm{Si}\,\sigma\left(x + \frac{D}{2}\right) - \mathrm{Si}\,\sigma\left(x - \frac{D}{2}\right)\right\}\right]. \end{aligned} \tag{38}$$

2.4.2. Off-centering of the Knife

Assuming that the spectral plane is unlimited in order to avoid too burdensome mathematical expressions, we will directly calculate the response obtained when the filter is transferred according to Ox by a quantity u_0.

The percussional response of the off-centered filter is, according to (24):

$$r(x) = -\frac{i}{\pi x} e^{-i2\pi u_0 x}.$$

For an object

$$f(x) = \mathrm{rect}\,\frac{x}{D}$$

where D represents the size of the field,

$$D = 2\sqrt{\frac{D_0^2}{4} - y^2}$$

(cf. §1-4) in the case of a circular field, we obtain the response:

$$R(x) = \frac{-i}{\pi} \int_{-\infty}^{+\infty} \mathrm{rect}\,\frac{x - z}{D}\, e^{-i2\pi u_0 z}\, dz \tag{39}$$

which, by integration, suggests

$$R(x) \doteq \frac{i}{\pi}\left[\mathrm{Ci}\,2\,\pi u_0\left|x+\frac{D}{2}\right| - \mathrm{Ci}\,2\,\pi u_0\left|x-\frac{D}{2}\right| + \right.$$

$$\left. + i\left\{\mathrm{Si}\,2\,\pi u_0\left(x-\frac{D}{2}\right) - \mathrm{Si}\,2\,\pi u_0\left(x+\frac{D}{2}\right)\right\}\right]. \tag{40}$$

To approximately one exponential factor (which disappears when the illuminations are considered) this expression is identical to (38). As a matter of fact, it is the same thing to transfer the filter with respect to the spectrum or the spectrum (centered on the geometric image of the source) with respect to the filter. A slant α of the wave corresponds to a transfer of the spectrum in spatial frequencies and the replacement of σ by $2\pi u_0$ in (38) does lead to (40). Figure 8 shows the illumination obtained for different penetrations of the phase knife.

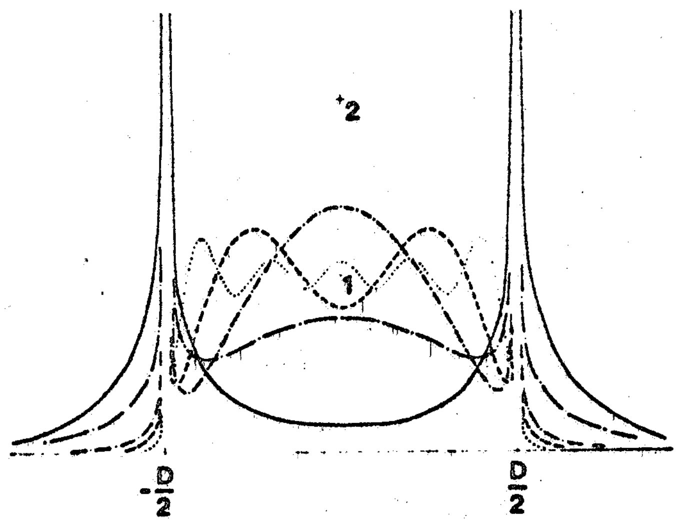


Figure 8. Off-centering of the Phase Knife by 0.2, 0.5, 1, 2 and 5 Halfwidths of the Diffraction Spot.

Case of the Foucault Method. It is only necessary to add rect x/D to the expression (40) to obtain the response in the case of the Foucault method. As a matter of fact, this answer is given by: /159

$$R_F(x) = \frac{1}{2}\,\mathrm{rect}\,\frac{x}{D} * e^{-i2\pi u_0 x}\,\delta(x) + \frac{1}{2}R(x)$$
$$= \frac{1}{2}\left[\mathrm{rect}\,\frac{x}{D} + R(x)\right]$$

considering the relation

$$f(x) * g(x)\,e^{-i2\pi u_0 x} = e^{i2\pi u_0 x}\left[f(x)\,e^{-i2\pi u_0 x} * g(x)\right]$$

therefore

$$\begin{aligned} R_F(x) &= \frac{1}{2}\,\mathrm{rect}\,\frac{x}{D} + \\ &+ \frac{i}{2\pi}\left[\mathrm{Ci}\,2\pi u_0\left|x + \frac{D}{2}\right| - \mathrm{Ci}\,2\pi u_0\left|x - \frac{D}{2}\right|\right. \\ &\left. + i\left\{\mathrm{Si}\,2\pi u_0\left(x - \frac{D}{2}\right) - \mathrm{Si}\,2\pi u_0\left(x + \frac{D}{2}\right)\right\}\right] \end{aligned} \tag{41}$$

which is the expression obtained by Linfoot [10].

2.5. Wide Source

In order to calculate the effect of the width of the source slit it is possible to use the general theory of partial coherency [8]. However, it is easier to use the results obtained in the previous paragraph.

By assuming that the source slit is illuminated in a completely incoherent way, it is possible, by simple integration, to calculate the illumination obtained in the case of a source slit with angular width α_0. The illumination in the observation plane is then given by:

$$E = \frac{1}{\alpha_0}\int_{-\alpha_0/2}^{+\alpha_0/2} E(x,\alpha)\,d\alpha \tag{42}$$

where $E(x,\alpha)$ is the expression given by (36).

The curves of Figure 9 represent the illumination obtained for various values of source width.

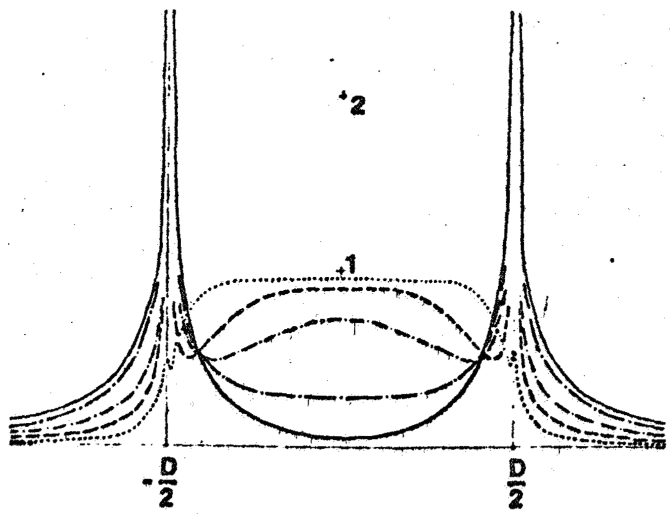


Figure 9. Effect of the Source Width. Illuminations obtained for the width of 0.2, 0.5, 1, 2 and 5 halfwidth of the diffraction spot.

Comment

The problem of an axial transfer of phase knife was not studied here, for it can be considered in two different ways. As a matter of fact, when the defocusing is considerable, the study can be carried out geometrically and we can return to the case of the "Minimumstrahlenzeichnung" [2, 15]. On the other hand, when the defocusing is slight, the diffraction must become a factor and we face the case of a slight focusing distortion. The study of this logically belongs within the larger context of the study of small aberrations, a chapter deserving special study.

3. Experimental Results

In this paragraph we shall provide some experimental findings confirming the preceding study. The experimental study was carried out using a very well polished mirror M working for a point close to its center (Figure 10).

The light source was a mercury vapor lamp, illuminating a thin slit F. A lens O set behind the phase shifting blade L formed the image of the mirror in plane Π where was arranged either a photographic plate, or a glass-fiber eyepiece connected to a photomultiplier and a plotting table allowing scanning of the field and measuring of illumination of any point.

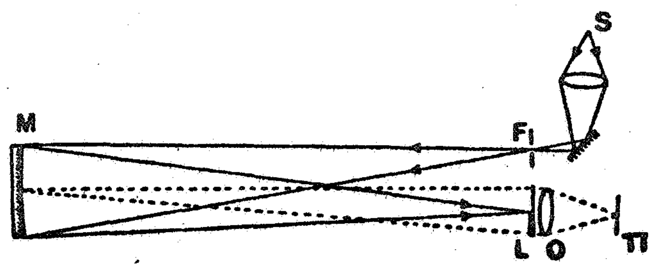


Figure 10. Experimental Setup.

Figure 11 shows the Hilbert transform of the uniformly illuminated mirror. Figure 12 shows the recording by means of a photomultiplier of the illumination following along one horizontal diameter of Figure 11. The height of the peaks is slightly less than that predicted by the theoretical study. However, this is, on one hand, owing to the integration performed by the glass fiber whose diameter is 72μ, and, on the other hand, to the width (not equal to zero) of the slit used. Figure 12b corresponds to the illumination observed in the absence of the phase shifting blade (it is indeed very important to check before any measurement whether or not the /160
field is uniformly lighted, otherwise the results are worthless).

In order to study the effect of phase shifting, we used a blade formed by a dielectric deposit produced in vacuo and whose thickness produced a phase shift less than π. By slanting the blade more or less, it was possible to achieve different phase shifts on both sides of the theoretical /161
value corresponding to the Hilbert transform. Figure 13 shows the results obtained for phase shifts varying between 164° and 185°. It can be seen that the dissymmetry changes side when the phase shift becomes greater than π. The result obtained is quite consistent with the theoretical study and shows that the filter must be made with precision, since a difference of 5° in the phase is clearly perceptible.

Figures 14, 15, and 16 confirm the theoretical results relative to the effect of off-centering and source width.

Figure 14 corresponds to the case of a very thin source, the knife being centered in the spectral plane.

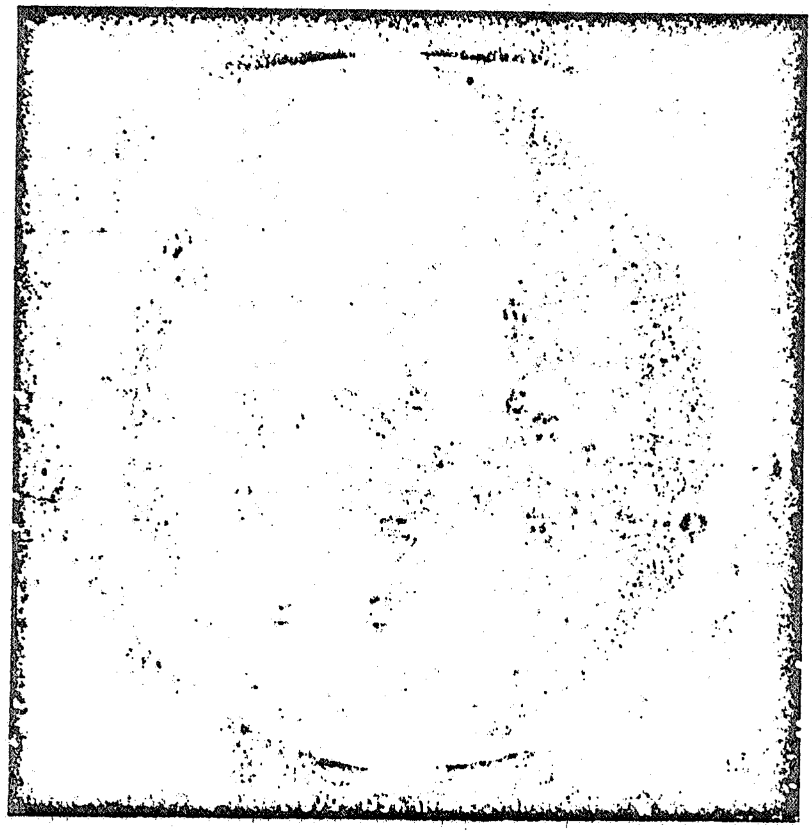


Figure 11. Hilbert Transform of a Plane and Homogeneous Wave.

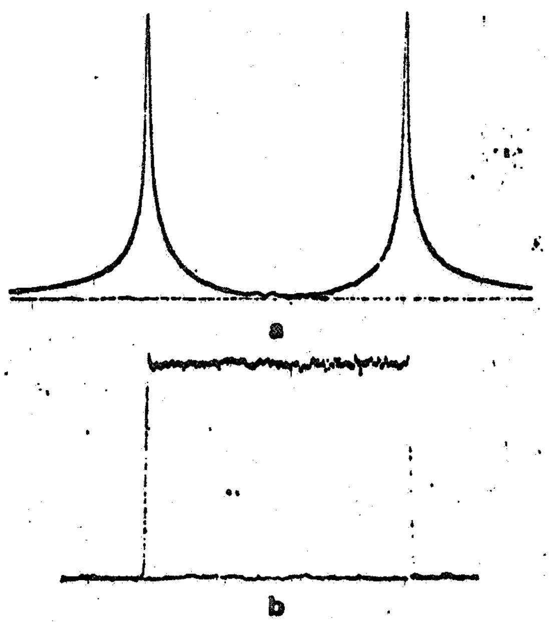


Figure 12. Photometric Scanning, a) of the Hilbert Transform, b) of the Image without the Phase Shifting Blade.

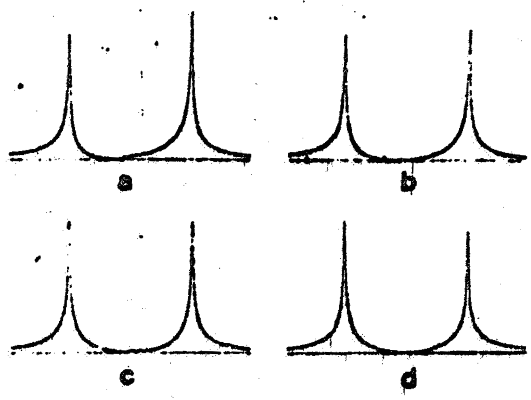


Figure 13. a) - b) Phase Shifting Below π, c) Equal to π, d) Above π.

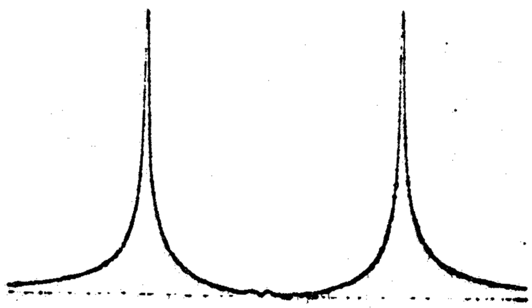


Figure 14. Hilbert Transform of a Spherical Wave for a Small Light Source Completely Centered.

Figures 15a, 15b, and 15c show repsectively the results obtained for off-centering the phase knife by 0.2, 1 and 10 times the halfwidth of the diffraction spot in the spectral plane. These results should be compared with the theoretical curves of Figure 8.

Figures 16a and 16b show the effect of the width of source for widths 0.2 and 0.5 times the diffraction half spot. Progressively as the width of the source increases, the center of the image is illuminated as predicted by the curves of Figure 9.

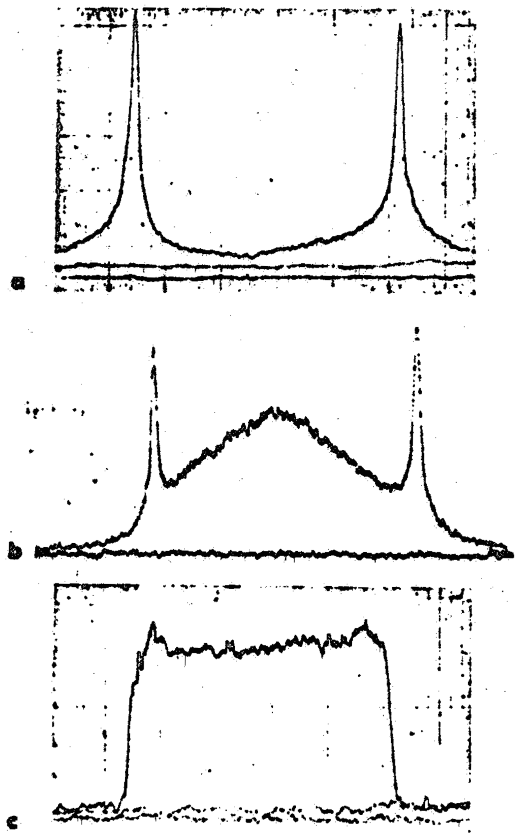


Figure 15. Off-centering of the Phase Knife by 0.2, 1 and 10 Times the Halfwidth of the Diffraction Spot.

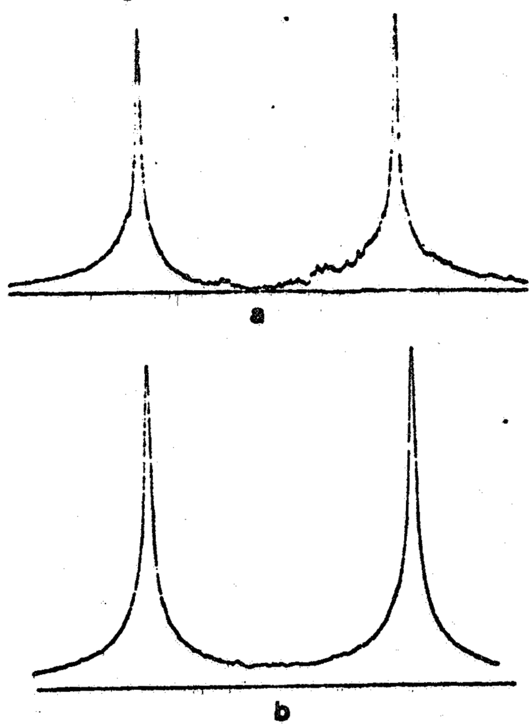


Figure 16. The Light Source Used has a Width 0.2 and 0.5 Times the Half Spot of the Diffraction.

Conclusion

After showing that it was possible, on a single basis, to solve the Hilbert transform in optics, it was found that the setup solving this transform contributed to the visualization of small phase distortions.

It was then endeavored to study the effect of different parameters which could modify the theoretical results. Indeed, it is not possible in practice to make filters, for example, which can exactly shift phase by 180°. Thus, it was possible to see that, when phase shifting has to be done with precision, a slight absorption is not harmful.

The effect of the position of the filter and width of light source was then studied. The experimental results obtained confirm the theoretical forecasts. A later study will endeavor to ascertain what advantages can be derived from this method for the observation of small phase distortions, particularly in the case of geometric aberrations.

REFERENCES

1. Francon, M., "Interferences by Double Diffraction in White Light," *Revue d'Optique*, Vol. 31, No. 2, pp. 65-86, 1951.
2. Wolter, H., *Schlieren, Phasenkontrast -- und Lichtschnittverfahren, Handbuch der Physik* [Schlieren, Phase Contrast -- and Light Slit Techniques, Handbook of Physics], Vol. XXIV, pp. 554-641, 1956.
3. Nomarski, G. and A. R. Weill, "Application to Metallography of Interferential Methods with Two Polarized Waves," *Rev. de Metallurgie*, L11, No. 2, p. 121.
4. Zernike, F., "Diffraction Theory of the Knife Edge Test and Its Improved Form in the Phase Contrast Method," M.N.R.A.S., Vol. 94, pp. 377-384, 1934.
5. Francon, M., "Phase Contrast In Optics and Microscopy," *Ed. Rev. Opt.*, Paris, 1950.
6. Lowenthal, S. and Y. Belvaux, "Observations of Phase Objects by Optically Processed Hilbert Transform," *Appl. Phys. Letters*, No. 2, pp. 49-51, 1967.
7. Marechal, A. and P. Croce, "A Frequency Filter for the Improvement of Contrast of Optical Images," *Reports Ac. Sc.*, No. 237, pp. 607-609, 1953.
8. Marechal, A. and M. Francon, "Diffraction, Structure if Images," *Ed. Rev. Opt.*, Paris, 1960.
9. Foucault, L., *Etude optique des surfaces* [Optical Study of Surfaces], Armand Colin, Paris, 1858.
10. Linfoot, E. H., *The Foucault Test, Recent Advances in Optics*, Clarendon Press, Oxford, pp. 128-174, 1950.
11. Yvon, G., "Examination of Optical Surfaces," *Ed. Rev. Opt.*, Paris, 1926.
12. Belvaux, Y. and S. Lowenthal, "White Light Hilbert Transform," *Appl. Optics*, Vol. 10, No. 3, pp. 676-677, 1971.
13. Belvaux, Y. and S. Lowenthal, "Birefringent Device Allowing Comparison of Foucault Method and Method of Hilbert Transform at Time of Observation of Phase Distortions," *Reports Ac. Sc.*, Vol. 269, No. 18, pp. 894-896, 1969.
14. *Tables of Integral Transform*, McGraw-Hill, New York, 1954.
15. Royer, H. and F. Albe, "A Method for Studying Very Small Phase Objects," *Reports Ac. Sc.*, No. 270, pp. 525-528, 1970.

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